Surface–atom force out of thermal equilibrium and its effect on ultra-cold atoms

Mauro Antezza

Dipartimento di Fisica, Università di Trento and CNR-INFM R&D Center on Bose–Einstein Condensation, Via Sommarive 14, I-38050 Povo, Trento, Italy

E-mail: antezza@science.unitn.it

Abstract
The surface–atom Casimir–Polder–Lifshitz force out of thermal equilibrium is investigated in the framework of macroscopic electrodynamics. Particular attention is devoted to its large distance limit that shows a new, stronger behaviour with respect to the equilibrium case. The frequency shift produced by the surface–atom force on the centre-of-mass oscillations of a harmonically trapped Bose–Einstein condensate and on the Bloch oscillations of an ultra-cold fermionic gas in an optical lattice are discussed for configurations out of thermal equilibrium.

PACS numbers: 03.75.Kk, 67.40.Db, 77.22.−d, 78.20.−e

1. Introduction

The electromagnetic force felt by a neutral atom near the surface of a substrate has been the object of intense investigation since the pioneering works by Casimir and Polder [1] and Lifshitz, Dzyaloshinskii and Pitaevskii [2, 3]. In addition to the fundamental character of the force, these studies [4] are presently motivated by the possibility of technological applications [5], by searching for stronger constrains on hypothetical non-Newtonian forces [6] as well as its role in biological systems [7].

New perspectives to study such a force are opened by the recent development in storing and manipulating ultra-cold atoms. Indeed experimental and theoretical research has been recently focused on the forces acting on ultra-cold atomic gases due to the presence of a nearby surface. They include atomic beams [8], Bose–Einstein condensates [9–11] and degenerate Fermi gases [12].

The surface–atom force at thermal equilibrium $F^{eq}(T, z)$ can be in general separated in two parts

$$F^{eq}(T, z) = F_0(z) + F_{th}^{eq}(T, z). \tag{1}$$

The first one, $F_0(z)$, is related to zero-point fluctuations ($T = 0$) of the electromagnetic field. At short distances z this force behaves like $1/z^4$ and is the analogue of the van der Waals–London inter-atomic force. At larger distances, the inclusion of relativistic retardation effects gives rise to the Casimir–Polder asymptotic behaviour [1, 3]

$$F_0(z)_{z\to\infty} = -\frac{3}{2}\frac{\hbar c\alpha_0}{\pi z^5}\frac{\varepsilon_0 - 1}{\varepsilon_0 + 1}\phi(\varepsilon_0), \qquad (2)$$

where α_0 and ε_0 are the static polarizability of the atom and the static dielectric function of the substrate, respectively. The function $\phi(\varepsilon_0) \sim 1$ is defined, for example, in [9]. The second contribution to the force, $F_{\text{th}}^{\text{eq}}(T, z)$, is due to the thermal fluctuations of the electromagnetic field. This contribution was first considered by Lifshitz [13] who applied the theory of electromagnetic fluctuations developed by Rytov [14]. At large distances the thermal contribution approaches the so-called Lifshitz law

$$F_{\text{th}}^{\text{eq}}(T, z)_{z\to\infty} = -\frac{3}{4}\frac{k_B T\alpha_0}{z^4}\frac{\varepsilon_0 - 1}{\varepsilon_0 + 1}. \qquad (3)$$

Such asymptotic behaviour is reached at distances larger than the thermal wavelength $\lambda_T = \hbar c/k_B T$, corresponding to $\sim$7.6 μm at room temperature. Thus, it is the leading contribution to the total force.

The Lifshitz force was originally evaluated at thermodynamic equilibrium. A non-trivial issue is the study of such a force out of thermal equilibrium, characterizing configurations where the temperature of the substrate T_S and the environment T_E, do not coincide. For instance in typical experiments with ultra-cold atomic gases, the environment temperature is determined by the chamber containing the substrate and the trapped atoms.

In this paper we describe the surface–atom force out of thermal equilibrium and how to recover its asymptotic behaviour at large distances. We assume that the radiation surrounding the atom is not able to populate its internal excited states which are assumed to be at energies $\hbar\omega_{at}$ much higher than the thermal energy

$$k_B T_S, \qquad k_B T_E \ll \hbar\omega_{at}. \qquad (4)$$

This condition is very well satisfied at ordinary temperatures (for example, the first optical resonance of Rb atoms corresponds to 1.8×10^4 K). In the last part of the paper, we analyse the effects of such a force on cold atoms, and in particular on the centre-of-mass motion of a trapped Bose–Einstein condensate and on the Bloch oscillations of ultra-cold fermionic atoms in an optical lattice.

2. The Green function formalism

In the calculation of the surface–atom force, the main ingredient is clearly the electromagnetic field and its sources. The latter, in our approach, is treated as point-like oscillating dipoles. Furthermore, it is useful to write the fields using Green's function formalism, Green's function being the solution of the wave equation for a point-like source. Once this solution is known, the solution due to a general source can be obtained by the principle of linear superposition. The dyadic Green function $\overline{\mathbf{G}}$ describing the electromagnetic field in surface optics (for isotropic, linear and non-magnetic media) is the solution of the equation

$$\nabla \wedge \nabla \wedge \overline{\mathbf{G}}[\omega; \mathbf{r}, \mathbf{r}'] - k^2\varepsilon(\omega; \mathbf{r})\overline{\mathbf{G}}[\omega; \mathbf{r}, \mathbf{r}'] = 4\pi k^2\overline{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}'), \qquad (5)$$

with the boundary conditions imposed by the geometry of the problem. In previous equation $k = \omega/c$ is the vacuum wavenumber, $\varepsilon(\omega; \mathbf{r})$ is the dielectric function and $\overline{\mathbf{I}}$ is the identity dyad. Equation (5) comes from the usual wave equation for the Fourier-transformed electric field

$$\nabla \wedge \nabla \wedge \mathbf{E}[\omega; \mathbf{r}] - k^2 \varepsilon(\omega; \mathbf{r})\mathbf{E}[\omega; \mathbf{r}] = 4\pi k^2 \mathbf{P}[\omega; \mathbf{r}], \tag{6}$$

obtained from the macroscopic Maxwell equations in which the sources are described by the effective electric polarization field $\mathbf{P}[\omega; \mathbf{r}]$ related to the electric current by $\mathbf{J}[\omega; \mathbf{r}] = -\mathrm{i}\omega\mathbf{P}[\omega; \mathbf{r}]$. The convolution of the solution obtained from equation (5) and the effective electric polarization gives the electric field at the observation point $\mathbf{r}$

$$\mathbf{E}[\omega; \mathbf{r}] = \int \overline{\mathbf{G}}[\omega; \mathbf{r}, \mathbf{r}'] \cdot \mathbf{P}[\omega; \mathbf{r}'] \, \mathrm{d}\mathbf{r}'. \tag{7}$$

3. Surface–atom force

Let us consider the atom described by its complex dielectric polarizability function $\alpha(\omega) = \alpha'(\omega) + \mathrm{i}\alpha''(\omega)$ in a vacuum half space V_1 and placed at a distance z from the surface of the dielectric half space V_2. Let us choose an orthogonal coordinate system with the xy plane coincident with the interface and the z axis such that the dielectric occupies the region with $z < 0$ and the vacuum of the region with $z > 0$. The force acting on a neutral atom without a permanent electric dipole moment is [15]

$$\mathbf{F}(\mathbf{r}) = \left\langle d_i^{\text{tot}}(t)\nabla' E_i^{\text{tot}}(\mathbf{r}', t)\big|_{\mathbf{r}} \right\rangle \approx \left\langle d_i^{\text{ind}}(t)\nabla' E_i^{\text{fl}}(\mathbf{r}', t)\big|_{\mathbf{r}} \right\rangle + \left\langle d_i^{\text{fl}}(t)\nabla' E_i^{\text{ind}}(\mathbf{r}', t)\big|_{\mathbf{r}} \right\rangle, \tag{8}$$

where d_i's are the atomic electric dipole components, we have used Einstein's summation convention for repeated indices and $\nabla' \equiv \nabla_{\mathbf{r}'}$. In equation (8), the average is done with respect to the state of the atom and of the field and the lowest order in perturbation theory has been considered. The first term describes the (spontaneous and thermal) field fluctuations correlated with the induced dipole, and the second term involves (spontaneous and thermal) dipole fluctuations correlated to the field they induce. The induced electric dipole for the atom at the position $\mathbf{r}$ is

$$\mathbf{d}^{\text{ind}}[\omega] = \alpha(\omega)\mathbf{E}^{\text{tot}}[\omega; \mathbf{r}] \approx \alpha(\omega)\mathbf{E}^{\text{fl}}[\omega; \mathbf{r}], \tag{9}$$

where $\mathbf{E}^{\text{fl}}[\omega; \mathbf{r}]$ is the fluctuating field, and now $\alpha(\omega)$ is the atomic polarizability of the atom in an unbounded space. By modelling the atom as a point-like source dipole $\mathbf{d}(t) = \mathbf{d}[\omega]\,\mathrm{e}^{-\mathrm{i}\omega t}$ at $\mathbf{r}$, the corresponding polarization in the frequency domain is $\mathbf{P}[\omega, \mathbf{r}''] = \mathbf{d}[\omega]\delta(\mathbf{r}'' - \mathbf{r})$, and the electric field at the position $\mathbf{r}'$ is

$$\mathbf{E}^{\text{ind}}[\omega; \mathbf{r}'] = \overline{\mathbf{G}}[\omega; \mathbf{r}', \mathbf{r}] \cdot \mathbf{d}^{\text{tot}}[\omega] \approx \overline{\mathbf{G}}[\omega; \mathbf{r}', \mathbf{r}] \cdot \mathbf{d}^{\text{fl}}[\omega]. \tag{10}$$

Using equations (9) and (10), the fluctuating dipole and field contributions to the surface–atom force (8) read

$$\left\langle d_i^{\text{ind}}(t)\nabla' E_j^{\text{fl}}(\mathbf{r}', t)\big|_{\mathbf{r}} \right\rangle = \iint \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}\omega'}{2\pi} \, \mathrm{e}^{-\mathrm{i}(\omega-\omega')t}\alpha(\omega)\nabla'\left\langle E_i^{\text{fl}}[\omega; \mathbf{r}] E_j^{\text{fl}\dagger}[\omega'; \mathbf{r}']\right\rangle\big|_{\mathbf{r}}, \tag{11}$$

$$\left\langle d_i^{\text{fl}}(t)\nabla' E_j^{\text{ind}}(\mathbf{r}', t)\big|_{\mathbf{r}} \right\rangle = \iint \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}\omega'}{2\pi} \, \mathrm{e}^{-\mathrm{i}(\omega-\omega')t}\nabla' G_{jk}^*[\omega; \mathbf{r}', \mathbf{r}]\big|_{\mathbf{r}}\left\langle d_i^{\text{fl}}[\omega] d_k^{\text{fl}\dagger}[\omega']\right\rangle, \tag{12}$$

where the integrations are over the whole real frequency axis.

4. Surface–atom force at thermal equilibrium

At thermal equilibrium, in order to calculate the average values in (11) and (12), it is possible to use the fluctuation dissipation theorem [14, 16]. One finds for the fluctuating dipoles

$$\langle d_i^{\mathrm{fl}}[\omega]d_j^{\mathrm{fl}\dagger}[\omega']\rangle = \frac{4\pi\hbar\delta(\omega - \omega')\delta_{ij}}{1 - \mathrm{e}^{-\hbar\omega/k_{\mathrm{B}}T}}\alpha''(\omega), \tag{13}$$

and for the fluctuating fields

$$\langle E_i^{\mathrm{fl}}[\omega; \mathbf{r}]E_j^{\mathrm{fl}\dagger}[\omega'; \mathbf{r}']\rangle = \frac{4\pi\hbar\delta(\omega - \omega')}{1 - \mathrm{e}^{-\hbar\omega/k_{\mathrm{B}}T}}\,\mathrm{Im}\,G_{ij}[\omega; \mathbf{r}, \mathbf{r}']. \tag{14}$$

After substituting the previous equalities into equations (11) and (12) and using the reciprocity theorem $G_{ij}[\omega; \mathbf{r}, \mathbf{r}'] = G_{ji}[\omega; \mathbf{r}', \mathbf{r}]$, the surface–atom force at thermal equilibrium becomes

$$F^{\mathrm{eq}}(T, z) = \frac{\hbar}{\pi}\int_0^\infty \mathrm{d}\omega \coth\left(\frac{\hbar\omega}{2k_{\mathrm{B}}T}\right)\mathrm{Im}[\alpha(\omega)\partial_z G_{ii}[\omega; \mathbf{r}, \mathbf{r}']|_{\mathbf{r}}]. \tag{15}$$

Because of the relation $\coth(\hbar\omega/2k_{\mathrm{B}}T) = 1 + 2\bar{n}(\omega/T)$, where $\bar{n}(\omega/T) = (\mathrm{e}^{\hbar\omega/k_{\mathrm{B}}T} - 1)^{-1}$ is the Bose factor, one can separate in equation (15) the zero-point fluctuations contribution $F_0(z)$ from the thermal contribution $F_{\mathrm{th}}^{\mathrm{eq}}(T, z)$. The latter term is the sum of two contributions arising from the two terms of equation (8). The first one is due to the field fluctuations and it is linear in α'. The second one arises from the dipole fluctuations and it is linear in α''. As long as the condition (4) is valid, the field fluctuations contribution is the leading term in $F_{\mathrm{th}}^{\mathrm{eq}}(T, z)$.

5. Surface–atom force out of thermal equilibrium

A first important investigation of the surface–atom force out of thermal equilibrium was carried out by Henkel *et al* [15] who calculated the force generated by a dielectric substrate at finite temperature by assuming that the environment temperature is zero. The principal motivation of that paper was the study of the force at short distances.

In this section we analyse the general case of an atom placed in a vacuum at a distance z from the flat surface of a substrate that we assume to be locally at thermal equilibrium at a temperature T_{S} which can be equal or different from the environment temperature T_{E}, the global system being in or out of thermal equilibrium, respectively, but in a stationary regime [17, 18]. In this configuration it is relatively easy to describe the radiation produced by the flat substrate, while it is less trivial to describe the radiation coming from the environment. To face this problem we use the Lifshitz trick [2] for which the vacuum half space is assumed to be a dielectric locally at thermal equilibrium with the temperature T_{E}, by introducing an infinitesimal imaginary part of its dielectric function. Using the fluctuation dissipation theorem and after integrating over an infinite volume, the vacuum half space produces a radiation corresponding to the one that in a real system is generated by the environment walls at T_{E}.

We refer to the substrate as to the half space 2 occupying the volume V_2 with $z < 0$, with dielectric function $\varepsilon_2(\omega) = \varepsilon_2'(\omega) + \mathrm{i}\varepsilon_2''(\omega)$ and in thermal equilibrium at the temperature T_{S}. The vacuum half space 1 instead occupies the volume V_1 with $z > 0$ and is characterized by a dielectric function $\varepsilon_1(\omega) = \varepsilon_1'(\omega) + \mathrm{i}\varepsilon_1''(\omega)$ and a temperature T_{E}. Only after calculating the electric fields in this configuration, we set $\varepsilon_1(\omega) = 1$.

As well as for the thermal equilibrium case, the surface–atom force out of thermal equilibrium can be written as

$$F^{\mathrm{neq}}(T_{\mathrm{S}}, T_{\mathrm{E}}, z) = F_0(z) + F_{\mathrm{th}}^{\mathrm{neq}}(T_{\mathrm{S}}, T_{\mathrm{E}}, z), \tag{16}$$

where the thermal contribution $F_{\text{th}}^{\text{neq}}(T_S, T_E, z)$, provided the condition (4) is satisfied, is dominated by the thermal part of the fluctuating fields correlation (11) only, as at thermal equilibrium[1].

The physical origin of the electromagnetic field is [14], the random fluctuating polarization field $\mathbf{P}[\omega; \mathbf{r}]$, whose correlations, at thermal equilibrium, are described by the fluctuation dissipation theorem

$$\langle P_k[\omega; \mathbf{r}] P_l^*[\omega'; \mathbf{r}'] \rangle = \frac{\delta(\omega - \omega')\delta(\mathbf{r} - \mathbf{r}')\delta_{kl}\hbar\varepsilon''(\omega)}{1 - e^{-\hbar\omega/k_B T}}. \tag{17}$$

Since the correlations of the source polarization field are *local*, the fluctuations of the sources at different points add up incoherently. Therefore, we can assume that in the whole space, the correlations of the sources are given by equation (17), valid for source dipoles in the two half-spaces *assumed to be locally* at thermal equilibrium at two different temperatures [20]. In order to calculate the field correlation function (11), we express the electromagnetic field in terms of its source polarization field via equation (7), and using equation (17) we write the thermal part of the surface–atom force out of thermal equilibrium as

$$F_{\text{th}}^{\text{neq}}(T_S, T_E, z) = F_{\text{th}}^{\text{neq}}(T_S, 0, z) + F_{\text{th}}^{\text{neq}}(0, T_E, z), \tag{18}$$

where the first thermal contribution

$$F_{\text{th}}^{\text{neq}}(T_S, 0, z) = \frac{\hbar}{2\pi^2} \int_0^\infty d\omega \frac{\varepsilon_2''(\omega) \operatorname{Re}\left[\alpha(\omega) \int_{V_2} G_{ik}[\omega; \mathbf{r}, \mathbf{r}']\partial_z G_{ik}^*[\omega; \mathbf{r}, \mathbf{r}'] d^3\mathbf{r}'\right]}{e^{\hbar\omega/k_B T_S} - 1} \tag{19}$$

arises from the sources in the substrate V_2, while the second one

$$F_{\text{th}}^{\text{neq}}(0, T_E, z) = \frac{\hbar}{2\pi^2} \int_0^\infty d\omega \frac{\varepsilon_1''(\omega) \operatorname{Re}\left[\alpha(\omega) \int_{V_1} G_{ik}[\omega; \mathbf{r}, \mathbf{r}']\partial_z G_{ik}^*[\omega; \mathbf{r}, \mathbf{r}'] d^3\mathbf{r}'\right]}{e^{\hbar\omega/k_B T_E} - 1} \tag{20}$$

is produced by the sources in the vacuum half space V_1.[2] It is possible to show that the sum of equations (19) and (20), at the same temperature, reproduce the thermal part of the force at thermal equilibrium [18]. Indeed it is possible to apply to the whole space Green's functions property (see, for example, [19])

$$\int_\Omega d\mathbf{r}\, \varepsilon''(\mathbf{r}, \omega) G_{ik}[\omega; \mathbf{r}_1, \mathbf{r}] G_{jk}^*[\omega; \mathbf{r}_2, \mathbf{r}] = 4\pi \operatorname{Im} G_{ij}[\omega; \mathbf{r}_1, \mathbf{r}_2], \tag{21}$$

where the integration is on the volume Ω such that on its surface the Green function is zero. Then we can express the complete surface–atom force out of thermal equilibrium in the convenient form

$$F^{\text{neq}}(T_S, T_E, z) = F^{\text{eq}}(T_E, z) + F_{\text{th}}^{\text{neq}}(T_S, 0, z) - F_{\text{th}}^{\text{neq}}(T_E, 0, z), \tag{22}$$

where the equilibrium force $F^{\text{eq}}(T, z)$ is given by (1) and $F_{\text{th}}^{\text{neq}}(T, 0, z)$ is defined by equation (19). Consistently with assumption (4), in deriving the thermal part of equation (22) we ignored terms proportional to the imaginary part of the atomic polarizability. For the same reason the *wind* contribution in equation (19) and (20), related to α'', can be ignored and the real part $\alpha'(\omega)$, corresponding to the *dispersive* contribution, can be replaced with its static

---

[1] It is worth noting that since zero-point fluctuations are not affected by condition (4), in the calculation of the zero temperature force $F_0(z)$ both dipole zero-point fluctuations (13) and field zero-point fluctuations (14) are needed.
[2] The Green function G_{ik} then reduces, respectively, to its transmitted component in equation (19) [21] and to its incident, reflected and local component in equation (20) [18].
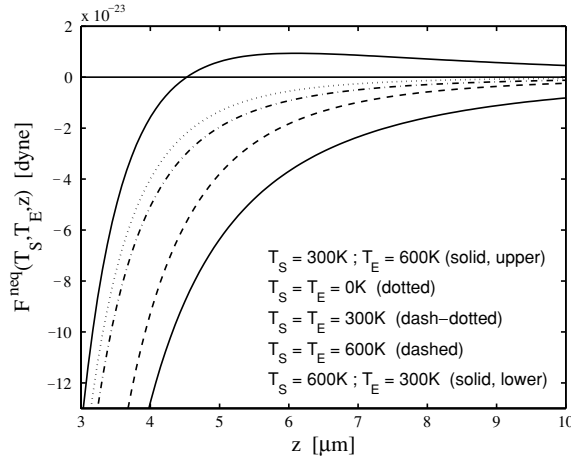
Figure 1. Surface–atom force $F^{neq}(z)$ calculated from equation (22), for different thermal configurations.

($\omega = 0$) value α_0. In this non-absorbing condition, the force of equation (18) can be also written in the form $F_{th}^{neq}(T_S, T_E, z) = 4\pi\alpha_0 \partial_z U_{El}(T_E, T_S, z)$, where $U_{El} = \langle E(z, t)^2 \rangle / 8\pi$ is the thermal component of the electric energy density at the atom position.

After some lengthy algebra we find for equation (19), the relevant result

$$F_{th}^{neq}(T, 0, z) = -\frac{2\sqrt{2}\hbar\alpha_0}{\pi c^4} \int_0^\infty d\omega \frac{\omega^4}{e^{\hbar\omega/k_B T} - 1} \int_1^\infty dq\, q \exp(-2z\sqrt{q^2 - 1}\,\omega/c)\sqrt{q^2 - 1}$$

$$\times \sqrt{|\varepsilon(\omega) - q^2| + (\varepsilon'(\omega) - q^2)} \left(\frac{1}{|\sqrt{\varepsilon(\omega) - q^2} + \sqrt{1 - q^2}|^2} \right.$$

$$\left. + \frac{(2q^2 - 1)(q^2 + |\varepsilon(\omega) - q^2|)}{|\sqrt{\varepsilon(\omega) - q^2} + \varepsilon(\omega)\sqrt{1 - q^2}|^2} \right), \tag{23}$$

where we introduced the dimensionless variable $q = Kc/\omega$, with K being the modulus of the electromagnetic wave-vector component parallel to the interface, and $\varepsilon(\omega) \equiv \varepsilon_2(\omega)$. In figure 1, we show the explicit results for the full force obtained from equation (22) as a function of the distance from the surface for different choices of T_S and T_E. The calculations have been performed for a sapphire substrate ($\varepsilon_0 = 9.41$) and for ^{87}Rb atoms ($\alpha_0 = 47.3 10^{-24}$ cm^3). For $F^{eq}(T, z)$, we have used the predictions of [9]. The figure clearly shows that the thermal effects out of equilibrium are sizable (solid lines), thereby providing promising perspectives for future measurements of the surface–atom force at large distances. To increase the attractive nature of the force it is much more convenient to heat the substrate by keeping the environment at room temperature (lower solid line) rather than heating the whole system (dashed line). When $T_S < T_E$ (upper solid line) the force exhibits a characteristic change of sign reflecting a repulsive nature at large distances (see also discussion below). At short distances the thermal correction to the force becomes smaller and smaller and is determined by the temperature of the substrate. The new effects are visible already at distances $z = 4$–7 μm, where experiments are now becoming available [10].

6. New asymptotic large distance limit

In this section we discuss in details the large z behaviour [17] of the out of equilibrium force (22). After the substitution $q^2 - 1 = t^2$, equation (23) becomes

$$F_{\text{th}}^{\text{neq}}(T, 0, z) = -\frac{2\sqrt{2}\hbar\alpha_0}{\pi c^4} \int_0^\infty d\omega \frac{\omega^4}{e^{\hbar\omega/k_B T} - 1} \int_0^\infty dt\, t^2\, e^{-2zt\omega/c} f(t, \omega), \qquad (24)$$

where

$$f(t, \omega) = \sqrt{|\varepsilon(\omega) - 1 - t^2| + (\varepsilon'(\omega) - 1 - t^2)} \left(\frac{1}{|\sqrt{\varepsilon(\omega) - 1 - t^2} + it|^2} \right.$$

$$\left. + \frac{(2t^2 + 1)(1 + t^2 + |\varepsilon(\omega) - 1 - t^2|)}{|\sqrt{\varepsilon(\omega) - 1 - t^2} + i\varepsilon(\omega)t|^2} \right). \qquad (25)$$

Due to the presence of the exponential $e^{-2zt\omega/c}$ in equation (24), it is possible to show that only the region $t \ll 1$ contributes to the large z behaviour of the force that in such a limit exhibits the non-trivial asymptotic behaviour

$$F_{\text{th}}^{\text{neq}}(T, 0, z)_{z\to\infty} = -\frac{\sqrt{2}\hbar\alpha_0}{z^3 2\pi c} \int_0^\infty d\omega \frac{\omega}{e^{\hbar\omega/k_B T} - 1} f(\omega). \qquad (26)$$

This force exhibits a slower $1/z^3$ decay with respect to the one holding at thermal equilibrium where it decays like $1/z^4$ (see equation (3)). In the above equation, we have introduced the low t expansion of equation (25)

$$f(\omega) = \sqrt{|\varepsilon(\omega) - 1| + [\varepsilon'(\omega) - 1]} \frac{2 + |\varepsilon(\omega) - 1|}{|\varepsilon(\omega) - 1|}. \qquad (27)$$

Results (26) and (27) provide the large distance behaviour ($z \to \infty$) of the force (23) where the only assumption made was condition (4). Due to the presence of the Bose factor, the force (26) depends on the optical properties of the substrate at frequencies of the order of $\sim k_B T/\hbar$.

For temperatures much smaller than $\hbar\omega_c/k_B$, where ω_c is the lowest characteristic frequency of the dielectric substrate, only the static value of the dielectric function is relevant and so we can replace $f(t, \omega)$ with its low frequency limit in equation (24). In this limit $f(t, \omega \to 0)$ is different from zero only for $0 < t < \sqrt{\varepsilon_0 - 1}$, and after the $t \ll 1$ expansion equation (24) becomes

$$F_{\text{th}}^{\text{neq}}(T, 0, z)_{z\to\infty} = -\frac{\hbar\alpha_0}{z^3 2\pi c} \frac{\varepsilon_0 + 1}{\sqrt{\varepsilon_0 - 1}} \int_0^\infty d\omega \frac{\omega}{e^{\hbar\omega/k_B T} - 1} \int_0^{2z\sqrt{\varepsilon_0 - 1}\omega/c} du\, u^2\, e^{-u}, \qquad (28)$$

where we performed the change of variable $u = 2zt\omega/c$ and replaced $f(t, \omega \to 0)$ with its $t \ll 1$ expansion $\sqrt{2}(\varepsilon_0 + 1)/\sqrt{\varepsilon_0 - 1}$. For

$$z \gg \frac{\lambda_T}{\sqrt{\varepsilon_0 - 1}}, \qquad (29)$$

where $\lambda_T = \hbar\omega/k_B T$ is the thermal wavelength, we can extend the upper limit of integration on u to $+\infty$ and so we obtain that the force (22) felt by the atom approaches the asymptotic behaviour

$$F^{\text{neq}}(T_S, T_E, z)_{z\to\infty} = -\frac{\pi}{6} \frac{\alpha_0 k_B^2 (T_S^2 - T_E^2)}{z^3 c\hbar} \frac{\varepsilon_0 + 1}{\sqrt{\varepsilon_0 - 1}}. \qquad (30)$$

Result (30) holds at low temperature with respect to the first dielectric function resonance ($T \ll \hbar\omega_c/k_B$) and at distances satisfying the condition (29) calculated at the relevant

temperatures T_S and T_E. Equation (30) shows that, at large distances, the new force is attractive or repulsive depending on whether the substrate temperature is higher or lower than the environment one. Furthermore, it exhibits a stronger temperature dependence with respect to the equilibrium force (3), explicitly contains the Planck constant and has a $1/z^3$ distance dependence[3].

The new dependence of $F^{neq}(T, 0, z)$ on temperature and distance can be physically understood by noting that the main contribution to the z-dependent part of the electric energy U_{El} arises from $t \ll 1$. Such values of t correspond to the component of the black-body radiation impinging on the surface from the dielectric side in a small interval of angles, of order of $(\lambda_T/z)^2$, near the angle of total reflection. This radiation creates slowly damping evanescent waves in vacuum. As a result $F^{neq}(T, 0, z)$ turns out to be, in accordance with equation (30), of order of $-(\alpha_0\lambda_T^2/z^3)U_{BB}$, where $U_{BB} \propto T^4$ is the energy density of the black-body radiation.

Equation (30) holds for a dielectric substrate where ε_0 is finite. If we want to find the large distance limit for a metal we should use equation (26). In the limit of small values of T we can use the Drude model. As only frequencies $\omega \sim k_B T/\hbar$ contribute, one can substitute in equation (27) $\varepsilon''(\omega) = 4\pi\sigma/\omega \gg 1$, the real part $\varepsilon'(\omega)$ remaining finite as $\omega \to 0$. Then one finds $f(\omega) \to \sqrt{\varepsilon''(\omega)} = 2\sqrt{\pi\sigma/\omega}$, where σ is the electric conductivity, so that for a Drude metal equation (30) is replaced by

$$F^{neq}(T_S, T_E, z)_{z\to\infty} = -\frac{\alpha_0\zeta(3/2)\sqrt{\sigma}k_B^{3/2}\left(T_S^{3/2} - T_E^{3/2}\right)}{z^3 c\sqrt{2\hbar}}, \tag{31}$$

where $\zeta(3/2) \sim 2.61$ is the Riemann function. It is easy to show that equation (31) is valid at the condition

$$z \gg \hbar^{3/2}c\sqrt{4\pi\sigma}/(k_B T)^{3/2}. \tag{32}$$

7. Effects of the surface–atom force on ultra-cold atoms

Ultra-cold gases can provide a useful probe of the surface–atom force. A *mechanical* tool sensitive to the gradient of the surface–atom force is in fact the frequency shift of the centre-of-mass oscillation of a trapped Bose–Einstein condensate [9, 10]. On the other hand, experiments based on Bloch oscillations are *interferometric* tools sensitive to the force itself [23, 12]. Finally, one could also think of interference experiments involving the macroscopic phase of Bose–Einstein condensates in a double-well potential [24, 25]. The position of the corresponding interference fringes are sensitive to the surface–atom potential. In the last part of this paper we discuss the first two above-mentioned experiments.

7.1. Effects on the collective oscillations of a trapped BEC

Bose–Einstein condensed gases [26] are very dilute, ultra-cold samples characterized by unique properties of coherence and superfluidity. The study of the collective oscillations [27] of a Bose–Einstein condensate provides a useful probe of the surface–atom potential. In fact, it is possible to measure with great accuracy the frequency of the centre-of-mass motion ω_{CM} of a condensate. For a harmonically trapped condensate, the frequency ω_{CM} corresponds to the harmonic trap frequency ω_z, where z is the direction of the oscillations. Thus, if a BEC in a harmonic trap is placed at a distance d from the surface of a substrate, the surface–atom

---

[3] Instead of calculating the asymptotic behaviour (30) of the force from the general equation (23), it is possible to produce a more direct derivation assuming from the very beginning that one can neglect *absorption* and dispersion of the dielectric function of the substrate [22].
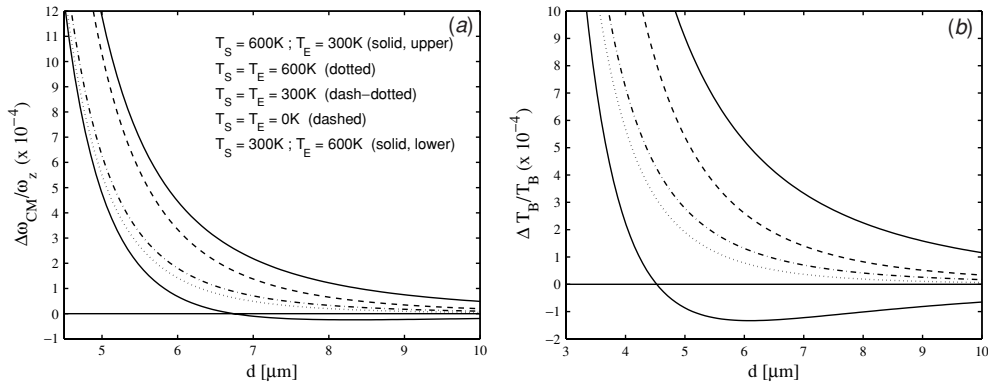
Figure 2. (a) Relative frequency shift (33) of the centre-of-mass oscillation of a BE condensate ($R_z = 2.5$ μm, $\omega_z/2\pi = 220$ Hz) and (b) relative shift of the Bloch oscillation period of a degenerate Fermi gas (34), out of thermal equilibrium (16).

potential $V_{s-a}(z)$ perturbs the trap potential and produces a shift in ω_{CM}. In the limit of small oscillations (for a complete analysis see [9]), such a shift is

$$\omega_{CM}^2 = \omega_z^2 + \frac{1}{m} \int_{-R_z}^{+R_z} dz\, n_0^z(z) \partial_z^2 V_{s-a}(z+d), \tag{33}$$

where $n_0^z(z)$ is a 1D column density of the gas (density integrated over the directions perpendicular to the direction of oscillation) and R_z is the Thomas–Fermi radius in the z direction[4]. Therefore, measuring ω_{CM} it is possible to extract the surface–atom potential $V_{s-a}(z)$ [9, 10].

In figure 2(a) we plotted, as a function of the surface–condensate separation d, the relative frequency shift $\Delta\omega_{CM}/\omega_z = (\omega_z - \omega_{CM})/\omega_z$ for the centre-of-mass oscillations of a ^{87}Rb condensate close to a sapphire substrate. In such a calculation we used the surface–atom potential corresponding to the force (22).

7.2. Effects on Bloch oscillations in Fermi gases

When an external force F_{ext} is applied to a particle trapped in a periodical potential, the particle undergoes oscillations in a momentum space (the Bloch oscillations). During this oscillations the particle quasi-momentum q evolves according to $\hbar\dot{q} = F_{ext}$. This is what happens for example in a sample of ultra-cold atoms trapped in a 1D optical lattice aligned along the vertical direction. Bloch oscillations produced by the effect of the gravity force $F_G = mg$ have a period $T_B = 4\pi\hbar/mg\lambda$, where λ is the lattice wavelength and g is the gravity acceleration. If now a surface is brought close to the atomic sample, the additional surface–atom force $F_{s-a}(z)$ perturbs the gravitational potential and affects the dynamics of the Bloch oscillations (for a complete analysis see [12, 23]). In particular, it produces a shift of the period T_B. In figure 2(b), we plotted the relative shift $\Delta T_B/T_B$ for different thermal configuration as the distance d between the centre of a cloud of ^{40}K fermionic atoms ($\alpha_0 = 4.3 \times 10^{-23}$ cm^3) and the surface of a sapphire substrate is varied. We also used the approximation of a small

---

[4] For a Bose–Einstein condensate in its ground state, the 1D column density is easily evaluated in the so-called Thomas–Fermi approximation where $n_0^z(z) = 15(1 - z^2/R_z^2)^2/16R_z$ [26].

cloud of Fermi atoms, for which

$$\frac{\Delta T_{\mathrm{B}}}{T_{\mathrm{B}}} = -\frac{F_{\mathrm{s-a}}(d)}{mg}.$$ (34)

In the range of distances plotted in figure 2(*b*), this approximation provides results in a good agreement with the exact calculation [12] that takes into account real experimental parameters of the gas.

It is worth noting that both effects of the surface–atom force out of thermal equilibrium described in the last section, and plotted in figures 2(*a*) and (*b*), are in the domain of the present experimental accuracy [10, 23].

Acknowledgments

References

[1] Casimir H B G and Polder D 1948 *Phys. Rev.* **73** 360
[2] Lifshitz E M 1956 *Zh. Eksp. Teor. Fiz.* **29** 94
    Lifshitz E M 1956 *Sov. Phys. JETP* **2** 73
[3] Dzyaloshinskii I E, Lifshitz E M and Pitaevskii L P 1961 *Adv. Phys.* **38** 165
[4] Babb J F *et al* 2004 *Phys. Rev.* A **70** 042901
    Caride A O *et al* 2005 *Phys. Rev.* A **71** 042901
[5] Chan H B *et al* 2001 *Phys. Rev. Lett.* **87** 211801
[6] Decca R S *et al* 2005 *Ann. Phys.* **318** 37–80
[7] Israelachvili J N 1991 *Intermolecular and Surface Forces* 2nd edn (San Diego: Academic)
    Gingell D and Fornes J A 1975 *Nature* **256** 210
[8] Sukenik C I *et al* 1993 *Phys. Rev. Lett.* **70** 560
    Shimizu F 2001 *Phys. Rev. Lett.* **86** 987
    Oberst H *et al* 2005 *Phys. Rev.* A **71** 052901
    Druzhinina V and DeKieviet M 2003 *Phys. Rev. Lett.* **91** 193202
[9] Antezza M, Pitaevskii L P and Stringari S 2004 *Phys. Rev.* A **70** 053619
[10] Harber D M, Obrecht J M, McGuirk J M and Cornell E A 2005 *Phys. Rev.* A **72** 033610
[11] Lin Y J *et al* 2004 *Phys. Rev. Lett.* **92** 050404
    Pasquini T A *et al* 2004 *Phys. Rev. Lett.* **93** 223201
[12] Carusotto I *et al* 2005 *Phys. Rev. Lett.* **95** 093202
[13] Lifshitz E M 1955 *Dokl. Akad. Nauk SSSR* **100** 879
[14] Rytov S M, Kravtsov Y A and Tatarskii V I 1989 *Principles of Statistical Radiophysics* vol 3 *Elements of Random Fields* (Berlin: Springer)
[15] Henkel C, Joulain K, Mulet J-P and Greffet J-J 2002 *J. Opt. A: Pure Appl. Opt.* **4** S109
[16] Lifshitz E M and Pitaevskii L P 1991 *Statistical Physics Part 2* (Oxford: Pergamon Press)
[17] Antezza M, Pitaevskii L P and Stringari S 2005 *Phys. Rev. Lett.* **95** 113202
[18] Antezza M 2006 in preparation
[19] Henry C H and Kazarinov R F 1996 *Rev. Mod. Phys.* **68** 801
[20] Polder D and Van Hove M 1971 *Phys. Rev.* B **4** 3303
[21] Henkel C, Joulain K, Carminati R and Greffet J J 2000 *Opt. Commun.* **186** 57
[22] Pitaevskii L P 2006 *J. Phys. A: Math. Gen.* **39** 6665
[23] Roati G *et al* 2004 *Phys. Rev. Lett.* **92** 230402
[24] Shin Y *et al* 2004 *Phys. Rev. Lett.* **92** 050405
[25] Collins L A *et al* 2005 *Phys. Rev.* A **71** 033628
[26] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 *Rev. Mod. Phys.* **71** 463
[27] Stringari S 1996 *Phys. Rev. Lett.* **77** 2360